Lecture 13: Procrustes, ICP

COMPSCI/MATH 290-04

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2/25/2016

Announcements

- Group Assignment 1 Full Submission Due next Tuesday 11:55 PM
- Hackathon Saturday 2/27 4:00 PM - 10:00 PM Gross Hall 330
- Rank Top 3 Final Project Choices By Next Wednesday 3/2 (Groups of 3-4)

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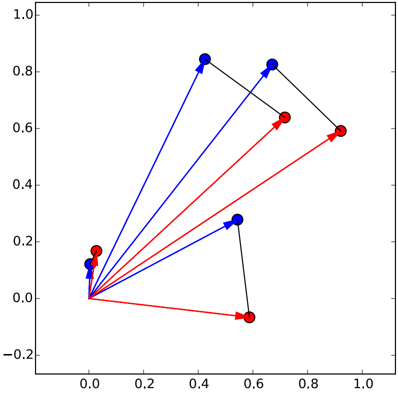
Procrustes Rotation Problem

Given correspondences, find best rotation

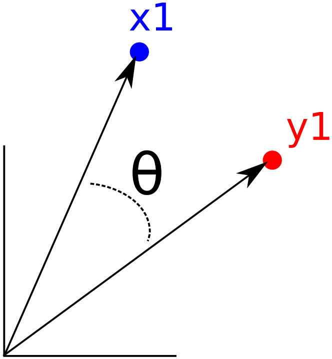


Rotating to Align Points: Dot Product Perspective

Points are well aligned if

$$\vec{x_1} \cdot \vec{y_1} = ||\vec{x_1}||||\vec{y_1}||\cos(\theta)$$

is *maximized*



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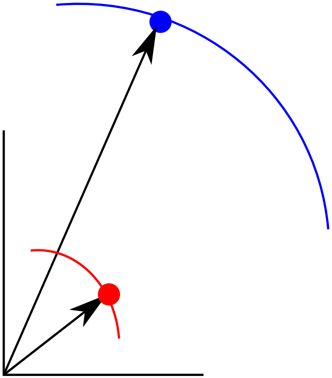
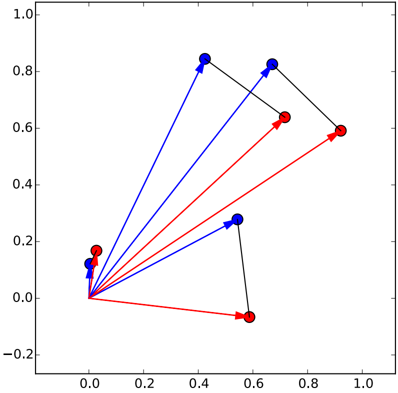


Rotating to Align Points: Dot Product Perspective

In general, how to maximize?

$$\sum_{i=1}^{N} R_x \vec{x}_i \cdot R_y \vec{y}_i$$



Rotating to Align Points: Dot Product Perspective

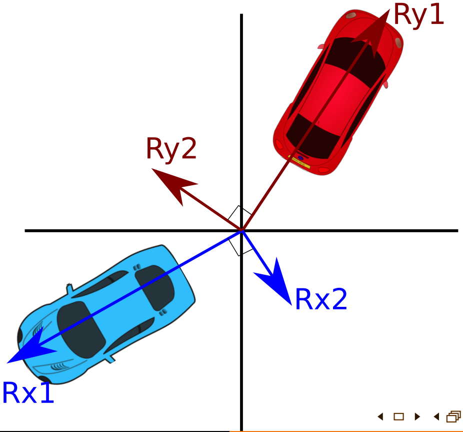
In general, how to maximize?

$$\sum_{i=1}^{N} R_x \vec{x}_i \cdot R_y \vec{y}_i$$

VIDEO EXAMPLE

Choosing Orthogonal Dot Product Axes

$$\sum_{i=1}^{N} \begin{bmatrix} - & \vec{R_{x1}} & - \\ - & \vec{R_{x2}} & - \\ - & \vec{R_{x3}} & - \end{bmatrix} \begin{bmatrix} | \\ \vec{x_i} \\ | \end{bmatrix} \cdot \begin{bmatrix} - & \vec{R_{y1}} & - \\ - & \vec{R_{y2}} & - \\ - & \vec{R_{y3}} & - \end{bmatrix} \begin{bmatrix} | \\ \vec{y_i} \\ | \end{bmatrix}$$



Choosing Orthogonal Dot Product Axes

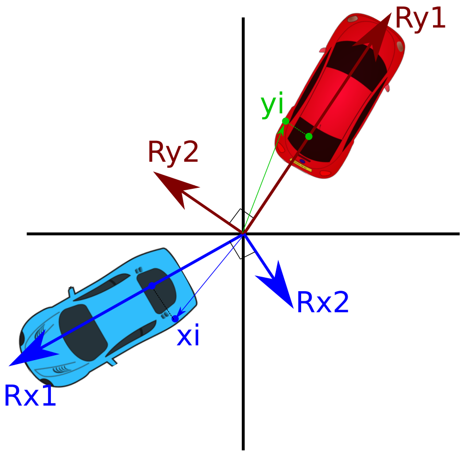
$$\sum_{i=1}^{N} \begin{bmatrix} - & \vec{R_{x1}} & - \\ - & \vec{R_{x2}} & - \\ - & \vec{R_{x3}} & - \end{bmatrix} \begin{bmatrix} | \\ \vec{x_i} \\ | \end{bmatrix} \cdot \begin{bmatrix} - & \vec{R_{y1}} & - \\ - & \vec{R_{y2}} & - \\ - & \vec{R_{y3}} & - \end{bmatrix} \begin{bmatrix} | \\ \vec{y_i} \\ | \end{bmatrix}$$

$$=$$

$$(\vec{R_{x1}} \cdot \vec{x_i})(\vec{R_{y1}} \cdot \vec{y_i}) + (\vec{R_{x2}} \cdot \vec{x_i})(\vec{R_{y2}} \cdot \vec{y_i}) + (\vec{R_{x3}} \cdot \vec{x_i})(\vec{R_{y3}} \cdot \vec{y_i})$$

Maximizing Dot Product: First Component

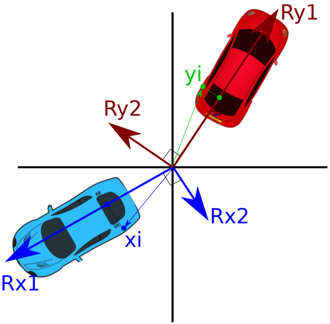
How to maximize $\sum_{i=1}^{N} (\vec{R_{x1}} \cdot \vec{x_i})(\vec{R_{y1}} \cdot \vec{y_i})$?



Maximizing Dot Product: First Component

$$R_{x1}X, R_{y1}Y$$

How to write $\sum_{i=1}^{N}(\vec{R_{x1}} \cdot \vec{x_i})(\vec{R_{y1}} \cdot \vec{y_i})$ in matrix form?

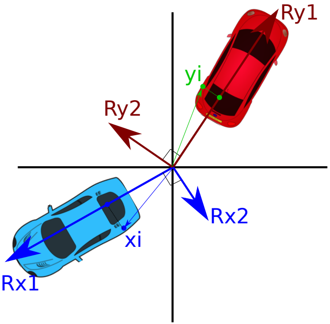


Maximizing Dot Product: First Component

$$R_{x1}X, R_{y1}Y$$

How to write $\sum_{i=1}^{N}(\vec{R_{x1}} \cdot \vec{x_i})(\vec{R_{y1}} \cdot \vec{y_i})$ in matrix form?

$$(R_{x1}X)(R_{y1}Y)^T = R_{x1}XY^TR_{y1}^T$$



Maximizing Dot Product: First Component

How to find u_1 and v_1 that maximize this product?

$$(R_{x1}X)(R_{y1}Y)^T = R_{x1}XY^TR_{y1}^T$$

Take SVD: $XY^T = USV^T$ and substitute in

$$R_{x1}USV^TR_{y1}^T$$

Maximizing Dot Product: First Component

$$R_{x1} U S V^T R_{y1}^T$$

$$\begin{bmatrix} - & \vec{R_{x1}} & - \end{bmatrix} \begin{bmatrix} | & | & | \\ \vec{u_1} & \vec{u_2} & \vec{u_3} \\ | & | & | \end{bmatrix} \begin{bmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & s_3 \end{bmatrix} \begin{bmatrix} - & \vec{v_1} & - \\ - & \vec{v_2} & - \\ - & \vec{v_3} & - \end{bmatrix} \begin{bmatrix} | \\ \vec{R_{y1}} \\ | \end{bmatrix}$$

Assume $s_1 > s_2 > s_3$, remember that U and V are orthogonal

Maximizing Dot Product: First Component

$$R_{x1} U S V^T R_{y1}^T$$

$$\begin{bmatrix} - & \vec{R_{x1}} & - \end{bmatrix} \begin{bmatrix} | & | & | \\ \vec{u_1} & \vec{u_2} & \vec{u_3} \\ | & | & | \end{bmatrix} \begin{bmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & s_3 \end{bmatrix} \begin{bmatrix} - & \vec{v_1} & - \\ - & \vec{v_2} & - \\ - & \vec{v_3} & - \end{bmatrix} \begin{bmatrix} | \\ \vec{R_{y1}} \\ | \end{bmatrix}$$

Assume $s_1 > s_2 > s_3$, remember that U and V are orthogonal
Raffle point question: Choose $\vec{R_{x1}}$ and $\vec{R_{y1}}$ to maximize this component of the dot product

Maximizing Dot Product: First Component

$$R_{x1} U S V^T R_{y1}^T$$

$$\begin{bmatrix} - & \vec{R_{x1}} & - \end{bmatrix} \begin{bmatrix} | & | & | \\ \vec{u_1} & \vec{u_2} & \vec{u_3} \\ | & | & | \end{bmatrix} \begin{bmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & s_3 \end{bmatrix} \begin{bmatrix} - & \vec{v_1} & - \\ - & \vec{v_2} & - \\ - & \vec{v_3} & - \end{bmatrix} \begin{bmatrix} | \\ \vec{R_{y1}} \\ | \end{bmatrix}$$

Assume $s_1 > s_2 > s_3$

$$\vec{R_{x1}} = \vec{u_1}, \vec{R_{y1}} = \vec{v_1}$$

In other words

- First row of R_x is first column of U
- First row of R_y is first column of V

Maximizing Dot Product: Second Component

What about the second rows of R_x and R_y for second component of dot product?

$$R_{x1} U S V^T R_{y1}^T$$

$$R_{x2} U S V^T R_{y2}^T$$

$$\begin{bmatrix} - & \vec{R_{x2}} & - \end{bmatrix} \begin{bmatrix} | & | & | \\ \vec{u_1} & \vec{u_2} & \vec{u_3} \\ | & | & | \end{bmatrix} \begin{bmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & s_3 \end{bmatrix} \begin{bmatrix} - & \vec{v_1} & - \\ - & \vec{v_2} & - \\ - & \vec{v_3} & - \end{bmatrix} \begin{bmatrix} | \\ \vec{R_{y2}} \\ | \end{bmatrix}$$

Assume $s_1 > s_2 > s_3$

$$\sum_{i=1}^{N} (\vec{R_{x1}} \cdot \vec{x_i})(\vec{R_{y1}} \cdot \vec{y_i})$$

Maximizing Dot Product: Full Rotation Matrix

$$\begin{bmatrix} - & \vec{u_1} & - \\ - & \vec{u_2} & - \\ - & \vec{u_3} & - \end{bmatrix} \begin{bmatrix} | & | & | \\ \vec{u_1} & \vec{u_2} & \vec{u_3} \\ | & | & | \end{bmatrix} \begin{bmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & s_3 \end{bmatrix} \begin{bmatrix} - & \vec{v_1} & - \\ - & \vec{v_2} & - \\ - & \vec{v_3} & - \end{bmatrix} \begin{bmatrix} | & | & | \\ \vec{v_1} & \vec{v_2} & \vec{v_3} \\ | & | & | \end{bmatrix}$$

$$R_x U S V^T R_y^T$$

Assume $s_1 > s_2 > s_3$

- Final answer for optimal rotations: $R_x = U^T, R^y = V^T$ (Carefully working with transposes)

Maximizing Dot Product: Full Rotation Matrix

- Final answer for optimal rotations: $R_x = U^T, R^y = V^T$

What if I want to just rotate Y and keep X fixed? What should R_y be?

Maximizing Dot Product: Full Rotation Matrix

- Final answer for optimal rotations: $R_x = U^T, R^y = V^T$

What if I want to just rotate Y and keep X fixed? What should R_y be?

$$R_y = UV^T$$

This is just SVD of XY^T without S !

Enforcing Right Handedness

$$\begin{bmatrix} - & \vec{u_1} & - \\ - & \vec{u_2} & - \\ - & \vec{u_3} & - \end{bmatrix} \begin{bmatrix} | & | & | \\ \vec{u_1} & \vec{u_2} & \pm\vec{u_3} \\ | & | & | \end{bmatrix} \begin{bmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & s_3 \end{bmatrix} \begin{bmatrix} - & \vec{v_1} & - \\ - & \vec{v_2} & - \\ - & \vec{v_3} & - \end{bmatrix} \begin{bmatrix} | \\ \vec{v_1} \\ | \end{bmatrix}$$

Check if $\vec{u_1} \times \vec{u_2} = \vec{u_3}$. If not, switch the sign of $\vec{u_3}$. This subtracts from the function we're trying to maximize, but it does the minimal damage since $s_3 < s_2 < s_1$

Rotation to Align Points: Algebra

$$\sum_{i=1}^{N} ||R\vec{x}_i - \vec{y}_i||_2^2$$

How is this the same thing as maximizing sum of dot products?

$$||R\vec{x} - \vec{y}||^2 = (R\vec{x} - \vec{y}) \cdot (R\vec{x} - \vec{y})$$

Rotation to Align Points: Algebra

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How is this the same thing as maximizing sum of dot products?

$$||R\vec{x} - \vec{y}||^2 = (R\vec{x} - \vec{y}) \cdot (R\vec{x} - \vec{y})$$

$$(R\vec{x} - \vec{y}) \cdot (R\vec{x} - \vec{y}) = R\vec{x} \cdot R\vec{x} + \vec{y} \cdot \vec{y} - 2R\vec{x} \cdot \vec{y}$$

Rotation to Align Points: Algebra

$$\sum_{i=1}^{N} ||R\vec{x}_i - \vec{y}_i||_2^2$$

How is this the same thing as maximizing sum of dot products?

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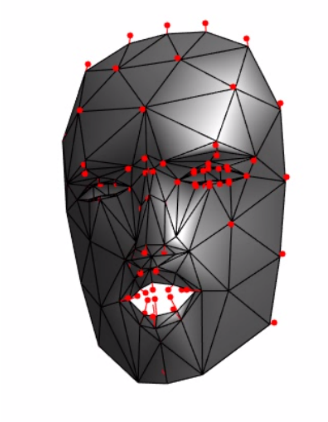
$$(R\vec{x} - \vec{y}) \cdot (R\vec{x} - \vec{y}) = R\vec{x} \cdot R\vec{x} + \vec{y} \cdot \vec{y} - 2R\vec{x} \cdot \vec{y}$$

$$(R\vec{x} - \vec{y}) \cdot (R\vec{x} - \vec{y}) = ||R\vec{x}||^2 + ||\vec{y}||^2 - 2R\vec{x} \cdot \vec{y}$$

Procrustes Application: Moving Head Alignment

VIDEO DEMO



Procrustes Application: Average Faces

Average Spaniards

https://pmsol3.wordpress.com/2011/04/07/world-of-averages-europeans/

Procrustes Application: Average Faces

Average Spaniards

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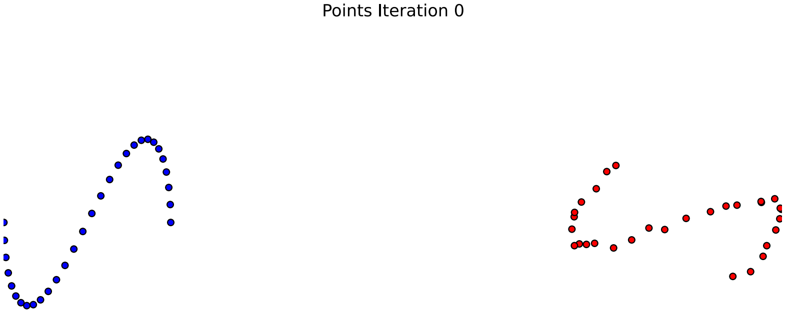
ICP

ICP Example

- Iterative Closest Points
- Iterative Closest Pairs
- Iterative Corresponding Points

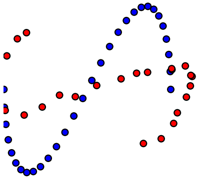
Goal: Automatically align two point sets



ICP Example

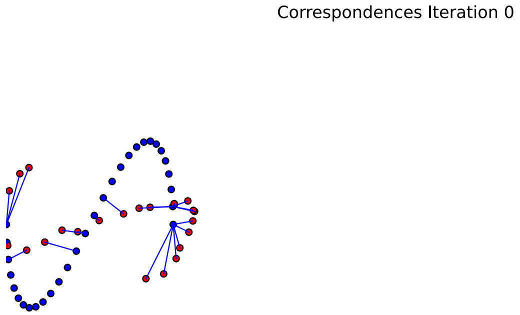
First mean center, now how to find correspondences automatically?

Mean-Center Iteration 0



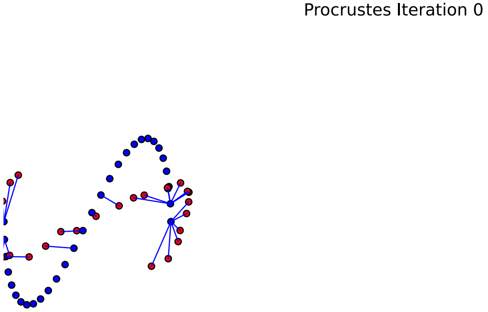
ICP Example

Use Nearest Neighbor, then do procrustes with those correspondences



Some blue points may be duplicated!

Use Nearest Neighbor, then do procrustes with those correspondences



Some points may be repeated!

ICP Example

Continue, interactive...

ICP Computation

Steps in ICP Loop

1. Find nearest neighbors
2. Procrustes rotation matrix
3. Rotate Y Points, go back to step 1

For N points in X and Y, what is time complexity of each step?

ICP Examples / Pitfalls

Video Examples

Final Projects

Choices

1. Equidecomposing polygon meshes into each other in 3D, with SLERP animation
2. Ghissi Alterpiece: Real Time Rendering Effects for NC Museum of Art
3. Nasher Muesum Brummer Statue Heads Speech Transfer
4. MOCAP Data Animation in Browser / Skinning / 3D Lemur Tracking(?)
5. 3D Face Verification

OR

6. Individual project with approval