Lecture 24: Heat Flow

COMPSCI/MATH 290-04

Chris Tralie, Duke University

4/12/2016

Announcements

- ▷ Group Assignment 3 Out: First Deadline Monday 4/18. Final Deadline Wednesday 4/27 (Sakai says 4/26 but that's wrong...e-mail me solution if go until 4/27)
- ▷ Final Project Final Deadline 5/3 5:00 PM

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Group Assignment 3 Preview

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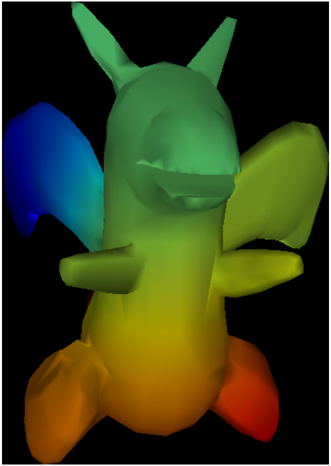
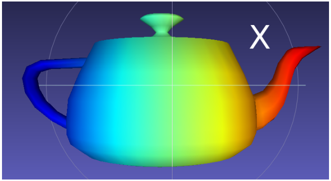
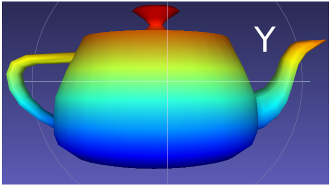
Mesh Scalar Fields (Functions)

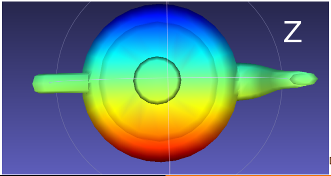


▷ A real number for every point on the surface

Coordinates As Functions

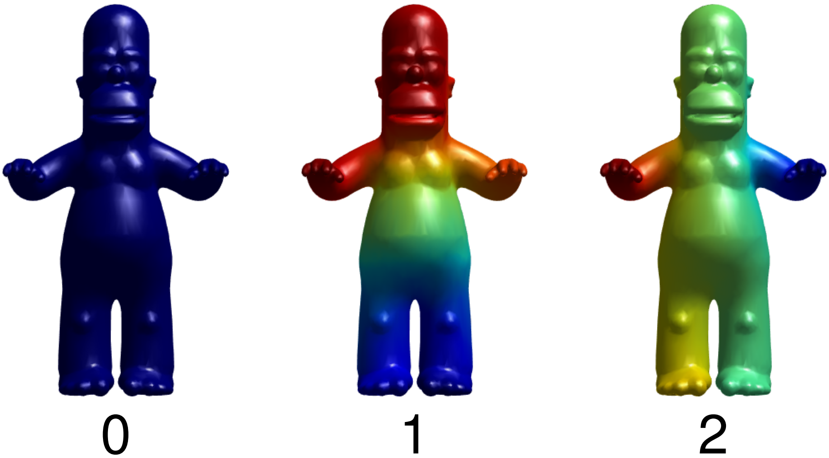




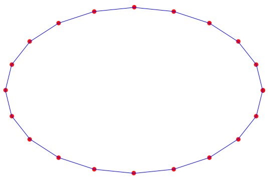


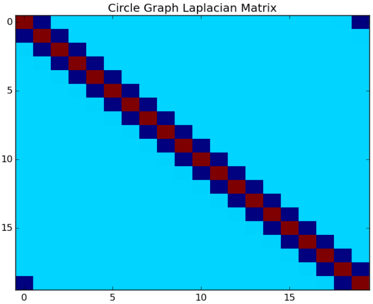
Laplacian Eigenfunctions (Homer Modes)

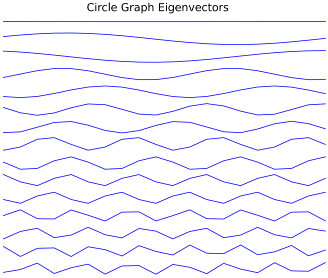
$$Lf = \lambda f$$



Discrete Circle Laplacian Eigenvectors

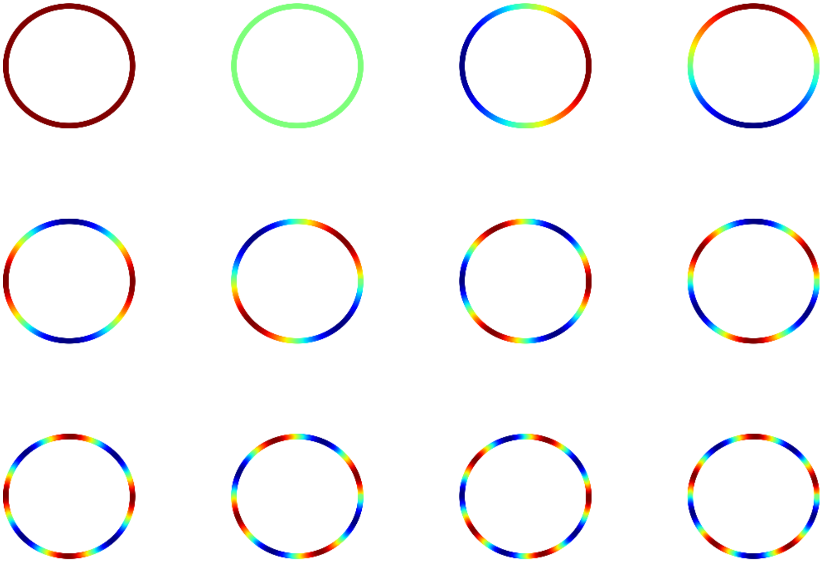






https://github.com/COMPSCI290-S2016/NumpyDemos/blob/master/1DLaplacian.py

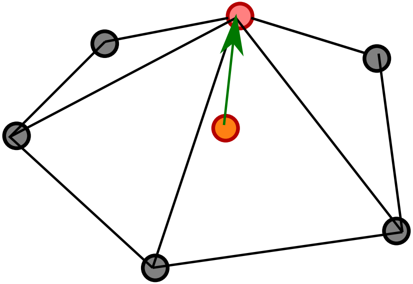
Discrete Circle Laplacian Eigenvectors



Curvature Vector

$$\delta_x = Lx, \delta_y = Ly, \delta_z = Lz$$

$$\delta = \sum_{j \in N(i)} (x_i, y_i, z_i) - (x_j, y_j, z_j)$$



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1D Heat Equation

Let $f(x,t)$ be the distribution of heat over a 1D bar of uniform material parameterized by x at time t. Then heat flow is governed by

$$\frac{\partial^2 f(x,t)}{\partial x^2} = \frac{\partial f(x,t)}{\partial t}$$

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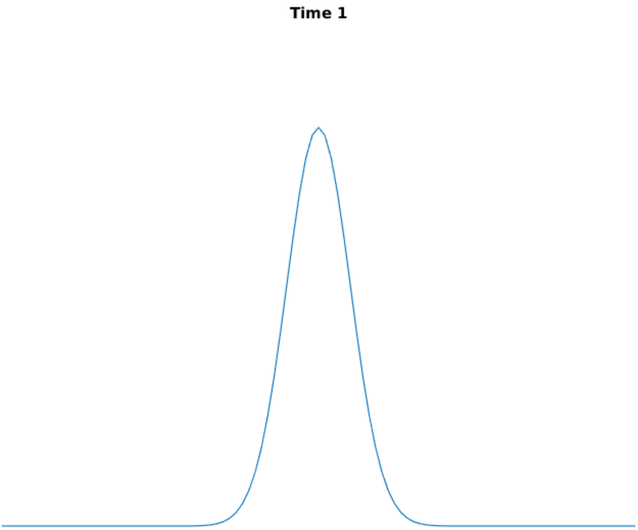
- The higher the curvature of the heat distribution, the faster it dissipates

1D Heat Flow Example



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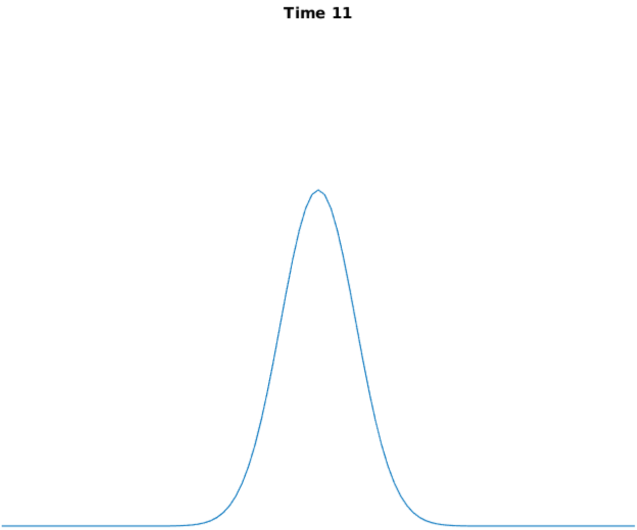

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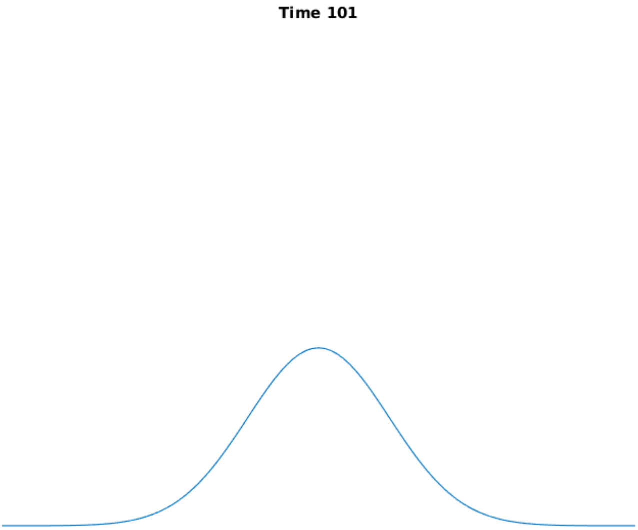

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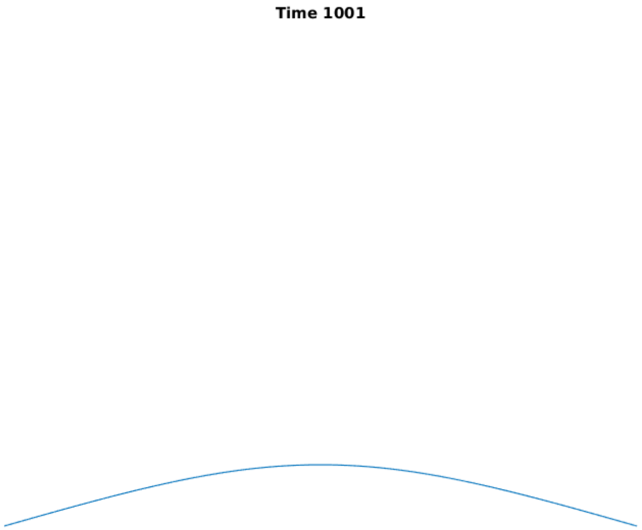

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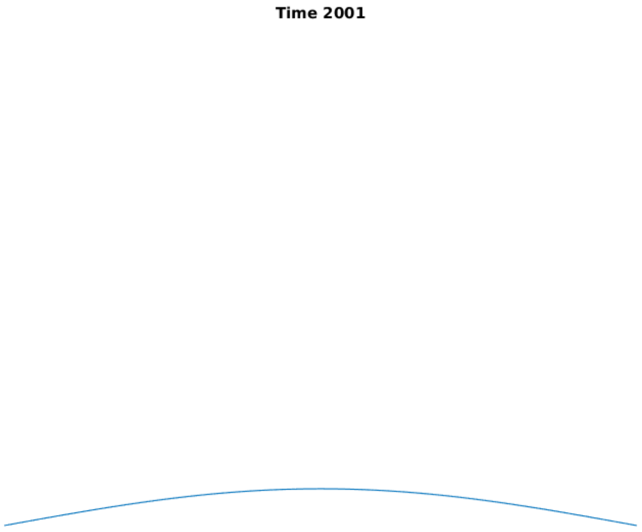

Generalizing To Surfaces

$$\frac{\partial^2 f(x,t)}{\partial x^2} = \frac{\partial f(x,t)}{\partial t}$$

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Video example...

Solutions To The Heat Equation

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Let ϕ_k be the k^{th} eigenvector of L and λ_k be the associated eigenvalue. Then solutions are

$$f_k = \phi_k e^{-\lambda t}$$

Initial Conditions

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- Project f_0 onto eigen basis, sum the solutions of each eigenvector individually

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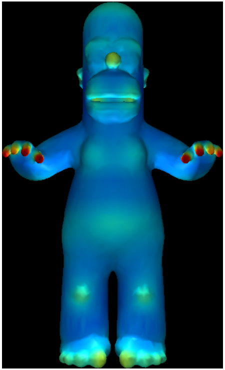
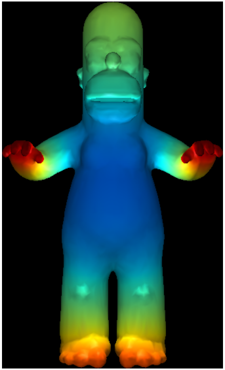
Demo...

Heat Kernel Signature

For every point, the fraction of heat that stays at that point after a certain amount of time

$t = 20$ $t = 500$





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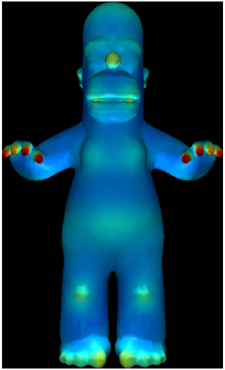
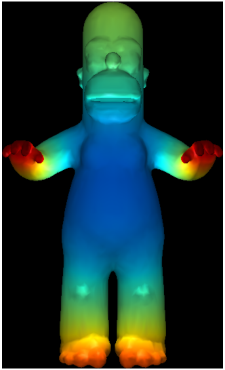
$t = 20$ 

$t = 500$ 

What does this look like? Multiscale curvature

Heat Kernel Signature: Computations

$$f(t)[i] = \sum_k (f_0^T \phi_k)[i] e^{-\lambda_k t} \phi_k[i]$$

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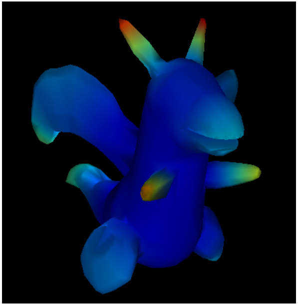
$$f(t)[i] = \sum_k e^{-\lambda_k t} \phi_k[i]^2$$

Another Example

$t = 20$



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