Lecture 5: Intro To Matrix Transformations

COMPSCI/MATH 290-04

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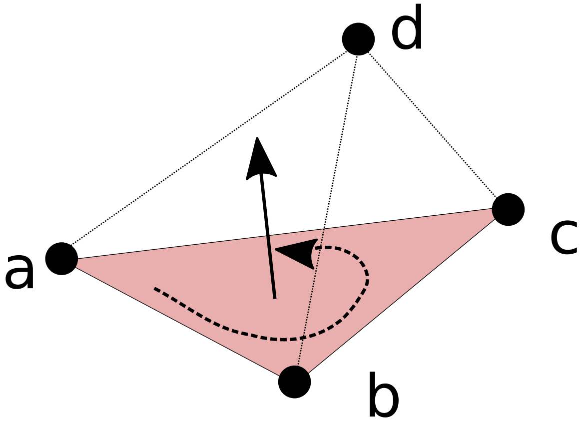
1/28/2016

Announcements

- Mini Assignment 1 Part 1 Graded
- Part 2 Due Friday 11:55 PM
  - Only 2D required
- Test Cases

Table of Contents

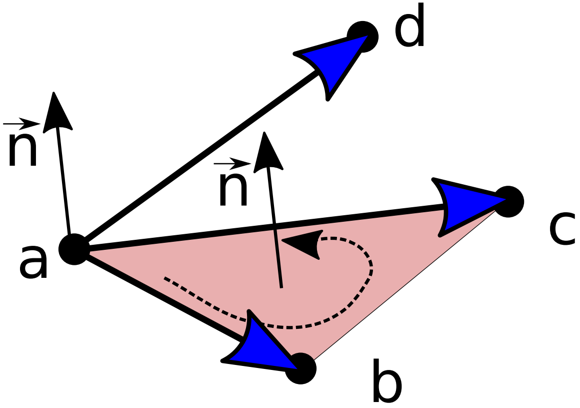
Point Above Plane: Right Hand Rule



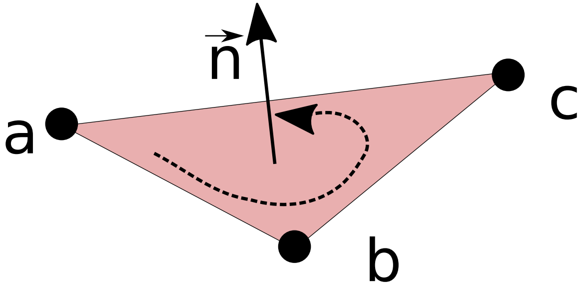
Point Above Plane: Right Hand Rule

$$\vec{n} = \vec{ab} \times \vec{ac}$$

$$\text{Test:}(\vec{ab} \times \vec{ac}) \cdot \vec{ad}$$



Right Hand Rule Perpendicular Bisector



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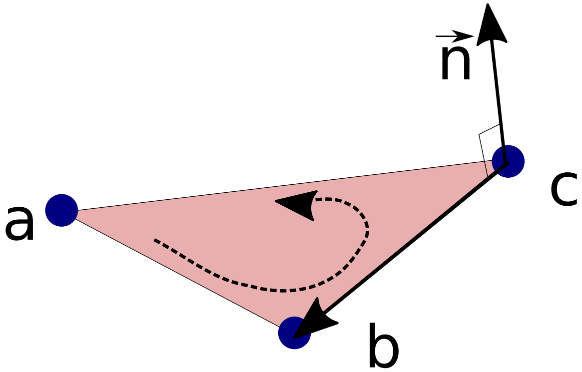


Right Hand Rule Perpendicular Bisector

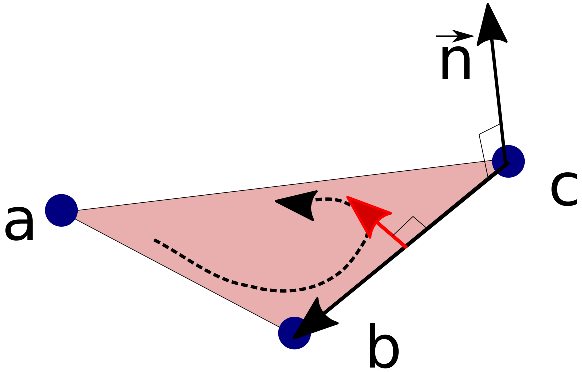


Table of Contents

Matrices And Matrix Multiplication

$$\begin{bmatrix} 3 & 4 \\ 3 & -3 \\ 4 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ -2 & 4 & 2 \end{bmatrix} = ?$$

Matrices And Matrix Multiplication

$$\begin{bmatrix} 3 & 4 \\ 3 & -3 \\ 4 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ -2 & 4 & 2 \end{bmatrix} = ?$$

$$AB_{i,j} = A_i \cdot B^j$$

Matrices And Matrix Multiplication

$$\begin{bmatrix} 3 & 4 \\ 3 & -3 \\ 4 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ -2 & 4 & 2 \end{bmatrix} = \begin{bmatrix} -5 & - & - \\ - & - & - \\ - & - & - \\ - & - & - \end{bmatrix}$$

$$\begin{bmatrix} 3 & 4 \\ 3 & -3 \\ 4 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ -2 & 4 & 2 \end{bmatrix} = \begin{bmatrix} -5 & 22 & - \\ - & - & - \\ - & - & - \\ - & - & - \end{bmatrix}$$

Matrices And Matrix Multiplication

$$\begin{bmatrix} 3 & 4 \\ 3 & -3 \\ 4 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ -2 & 4 & 2 \end{bmatrix} = \begin{bmatrix} -5 & 22 & 17 \\ - & - & - \\ - & - & - \\ - & - & - \end{bmatrix}$$

$$\begin{bmatrix} 3 & 4 \\ 3 & -3 \\ 4 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ -2 & 4 & 2 \end{bmatrix} = \begin{bmatrix} -5 & 22 & 17 \\ 9 & - & - \\ - & - & - \\ - & - & - \end{bmatrix}$$

Matrices And Matrix Multiplication

$$\begin{bmatrix} 3 & 4 \\ \color{blue}{3} & \color{blue}{-3} \\ 4 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & \color{blue}{2} & 3 \\ -2 & \color{blue}{4} & 2 \end{bmatrix} = \begin{bmatrix} -5 & 22 & 17 \\ 9 & \color{blue}{-6} & - \\ - & - & - \\ - & - & - \end{bmatrix}$$

Matrices And Matrix Multiplication

$$\begin{bmatrix} 3 & 4 \\ \color{blue}{3} & \color{blue}{-3} \\ 4 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & \color{blue}{3} \\ -2 & 4 & \color{blue}{2} \end{bmatrix} = \begin{bmatrix} -5 & 22 & 17 \\ 9 & -6 & \color{blue}{3} \\ - & - & - \\ - & - & - \end{bmatrix}$$

Matrices And Matrix Multiplication

$$\begin{bmatrix} 3 & 4 \\ 3 & -3 \\ 4 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ -2 & 4 & 2 \end{bmatrix} = \begin{bmatrix} -5 & 22 & 17 \\ 9 & -6 & 3 \\ 6 & - & - \\ - & - & - \end{bmatrix}$$

Matrices And Matrix Multiplication

$$\begin{bmatrix} 3 & 4 \\ 3 & -3 \\ 4 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ -2 & 4 & 2 \end{bmatrix} = \begin{bmatrix} -5 & 22 & 17 \\ 9 & -6 & 3 \\ 6 & 4 & - \\ - & - & - \end{bmatrix}$$

$$\begin{bmatrix} 3 & 4 \\ 3 & -3 \\ 4 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ -2 & 4 & 2 \end{bmatrix} = \begin{bmatrix} -5 & 22 & 17 \\ 9 & -6 & 3 \\ 6 & 4 & 10 \\ - & - & - \end{bmatrix}$$

$$\begin{bmatrix} 3 & 4 \\ 3 & -3 \\ 4 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ -2 & 4 & 2 \end{bmatrix} = \begin{bmatrix} -5 & 22 & 17 \\ 9 & -6 & 3 \\ 6 & 4 & 10 \\ -4 & - & - \end{bmatrix}$$

Matrices And Matrix Multiplication

$$\begin{bmatrix} 3 & 4 \\ 3 & -3 \\ 4 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ -2 & 4 & 2 \end{bmatrix} = \begin{bmatrix} -5 & 22 & 17 \\ 9 & -6 & 3 \\ 6 & 4 & 10 \\ -4 & 8 & - \end{bmatrix}$$

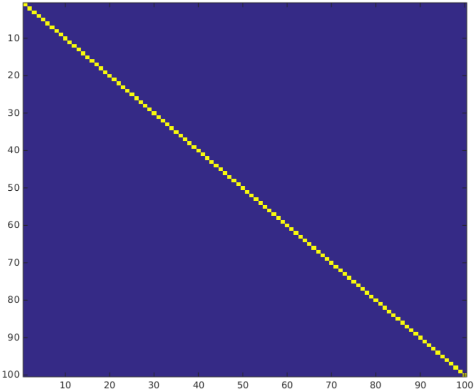
$$\begin{bmatrix} 3 & 4 \\ 3 & -3 \\ 4 & -1 \\ \color{blue}{0} & \color{blue}{2} \end{bmatrix} \begin{bmatrix} 1 & 2 & \color{blue}{3} \\ -2 & 4 & \color{blue}{2} \end{bmatrix} = \begin{bmatrix} -5 & 22 & 17 \\ 9 & -6 & 3 \\ 6 & 4 & 10 \\ -4 & 8 & \color{blue}{4} \end{bmatrix}$$

Matrix Multiplication Observations

- An $M \times K$ matrix time a $K \times N$ matrix is an $M \times N$ matrix
  - Otherwise undefined
- Matrix multiplication as defined takes $(M \times N \times K)$ time
  - $O(N^3)$. Fastest known algorithm is $O(N^{2.3728639})$

Identity Matrix

$$I_{i,j} = \left\{ \begin{array}{ll} 1 & i = j \\ 0 & \text{otherwise} \end{array} \right\}$$



Identity Matrix

Left-Handed Identity: $IA = A$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 3 & -3 \\ 4 & -1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 3 & -3 \\ 4 & -1 \\ 0 & 2 \end{bmatrix}$$

Right-Handed Identity: $AI = A$

$$\begin{bmatrix} 3 & 4 \\ 3 & -3 \\ 4 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 3 & -3 \\ 4 & -1 \\ 0 & 2 \end{bmatrix}$$

What Is This Really Doing?

$$\begin{bmatrix} 3 & 4 \\ 3 & -3 \\ 4 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x + 4y \\ 3x - 3y \\ 4x - y \\ 2y \end{bmatrix}$$

An $M \times N$ matrix is a *linear function* from $\mathbb{R}^N$ to $\mathbb{R}^M$

Linear Functions

$$f(ax + by) = af(x) + bf(y)$$

For matrices

$$A(aX + bY) = aAX + bAY$$

Every linear function can be written in matrix form

Nonlinear Function Example

$$f(x, y) = x^2 + y^2$$

$$\mathbb{R}^2 \to \mathbb{R}$$

Nonlinear Function Example

$$f(x, y) = x^2 + y^2$$

$$\mathbb{R}^2 \rightarrow \mathbb{R}$$

$$f(ax_1, ay_1) + f(bx_2, by_2) = a^2x_1^2 + a^2y_1^2 + b^2x_2^2 + b^2y_2^2$$

Nonlinear Function Example

$$f(x, y) = x^2 + y^2$$

$$\mathbb{R}^2 \to \mathbb{R}$$

$$f(ax_1, ay_1) + f(bx_2, by_2) = a^2x_1^2 + a^2y_1^2 + b^2x_2^2 + b^2y_2^2$$

$$f(ax_1 + bx_2, ay_1 + by_2) = a^2x_1^2 + a^2y_1^2 + b^2x_2^2 + b^2y_2^2 +$$

$$2abx_1x_2 + 2aby_1y_2$$

Nonlinear Function Example (!)

$$f(x) = x + 2$$

$$\mathbb{R} \to \mathbb{R}$$

Nonlinear Function Example (!)

$$f(x) = x + 2$$

$$\mathbb{R} \rightarrow \mathbb{R}$$

$$\begin{aligned} f(ax) + f(by) &= ax + 2 + bx + 2 && (1) \\ &= ax + bx + 4 && (2) \end{aligned}$$

Nonlinear Function Example (!)

$$f(x) = x + 2$$

$$\mathbb{R} \to \mathbb{R}$$

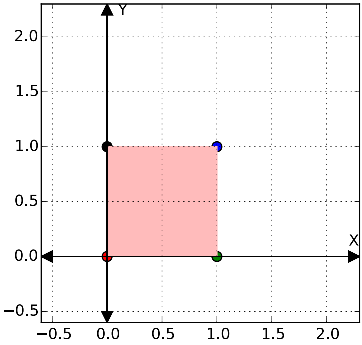
$$f(ax) + f(by) = ax + 2 + bx + 2 \quad (1)$$
$$= ax + bx + 4 \quad (2)$$

$$f(ax + by) = ax + by + 2$$

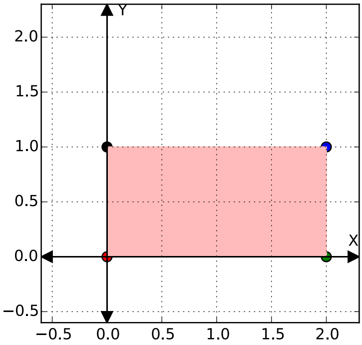
Table of Contents

2D Scale X

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ y \end{bmatrix}$$



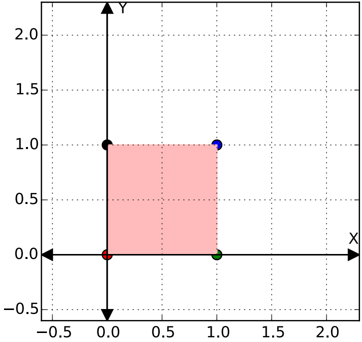
Before



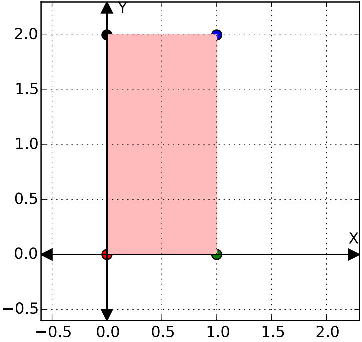
After

2D Scale Y

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 2y \end{bmatrix}$$



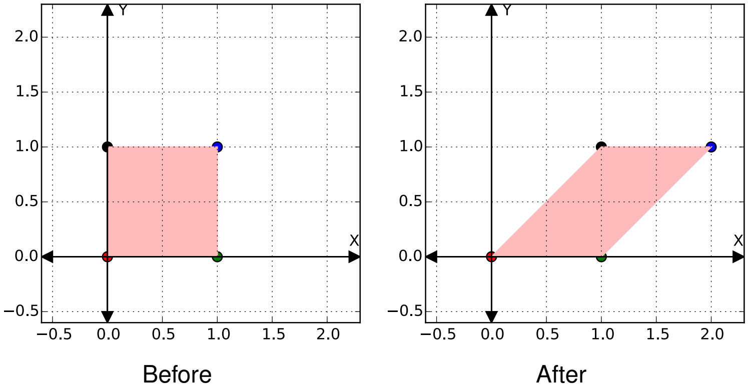
Before



After

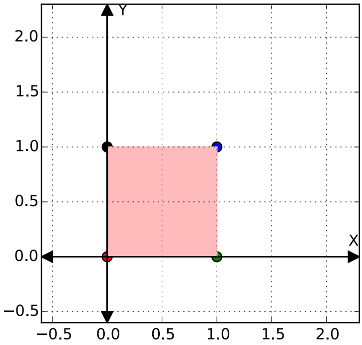
2D Shear X

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + y \\ y \end{bmatrix}$$

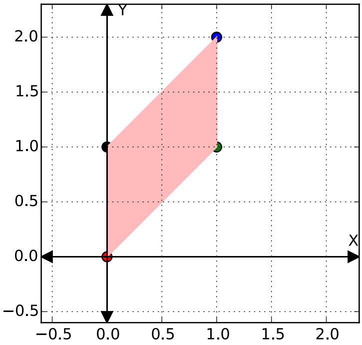


2D Shear Y

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ x + y \end{bmatrix}$$



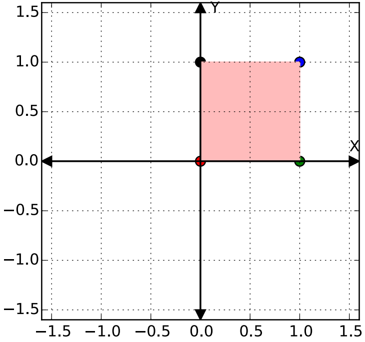
Before



After

2D Flip X

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}$$

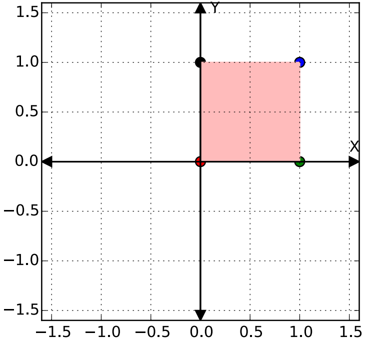


Before After

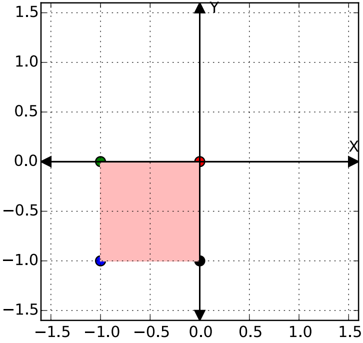
2D Flip X And Y

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ -y \end{bmatrix}$$

(actually a rotation by π about the origin)



Before



After

Matrix Compositions

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Matrix Compositions

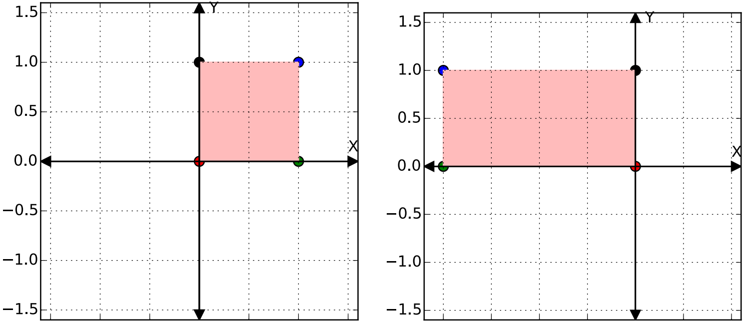
$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \right)$$

Matrix Compositions

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \right)$$

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2x \\ y \end{bmatrix} = \begin{bmatrix} -2x \\ y \end{bmatrix}$$

Scale, then flip



Matrix Compositions: Associative Rule

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Matrix Compositions: Associative Rule

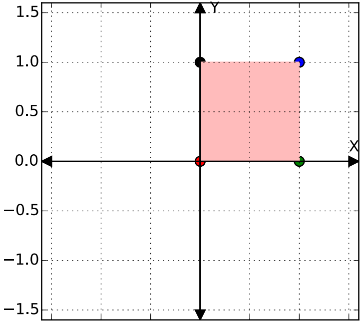
$$\left(\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x \\ y \end{bmatrix}$$

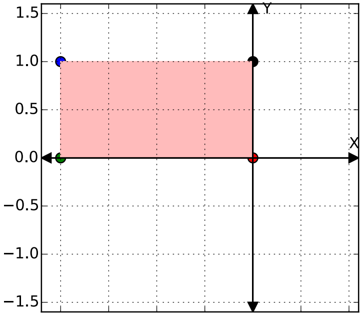
Matrix Compositions: Associative Rule

$$\left(\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2x \\ y \end{bmatrix}$$

Scale, then flip





Matrix Compositions: Commutative?

Flip, then scale?

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Matrix Compositions: Commutative?

Flip, then scale?

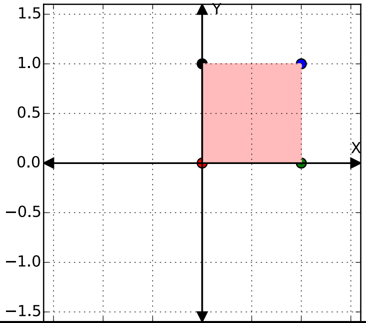
$$\left(\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix}$$

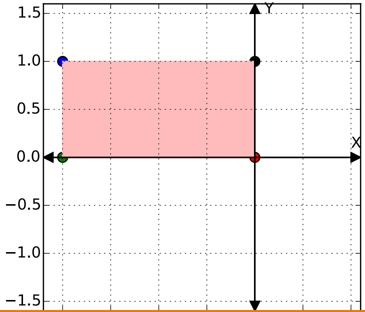
Matrix Compositions: Commutative?

Flip, then scale?

$$\left(\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2x \\ y \end{bmatrix}$$





Matrix Compositions: Commutative?

Skew, then flip

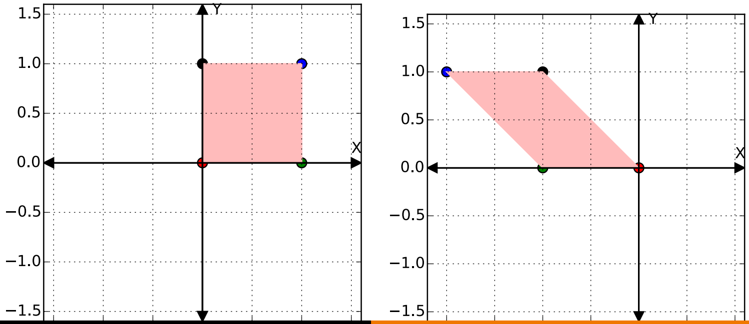
$$\left(\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix}$$

Matrix Compositions: Commutative?

Skew, then flip

$$\left(\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x-y \\ y \end{bmatrix}$$



Matrix Compositions: Commutative?

Flip, then skew

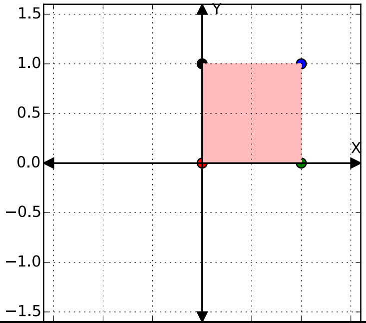
$$\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix}$$

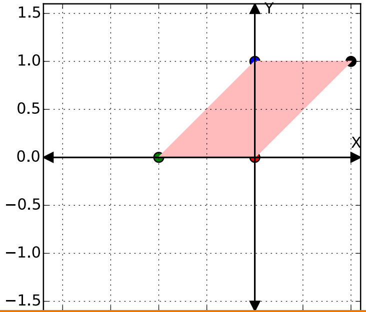
Matrix Compositions: Commutative?

Flip, then skew

$$\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x + y \\ y \end{bmatrix}$$





Commutativity Conclusion

In general, matrix multiplication does not commute!

Table of Contents

Column Vector Walking Interpretation

$$\begin{bmatrix} | & | & \vdots & | & \vdots & | \\ \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_k & \dots & \vec{v}_N \\ | & | & \vdots & | & \vdots & | \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ a_k \\ \vdots \\ 0 \end{bmatrix} = a_k \vec{v}_k$$

Column Vector Walking Interpretation

$$\begin{bmatrix} | & \vdots & | & \vdots & | & \vdots & | \\ \vec{v_1} & \dots & \vec{v_k} & \dots & \vec{v_j} & \dots & \vec{v_N} \\ | & \vdots & | & \vdots & | & \vdots & | \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ a_k \\ \vdots \\ a_j \\ \vdots \\ 0 \end{bmatrix} = a_k \vec{v_k} + a_j \vec{v_j}$$

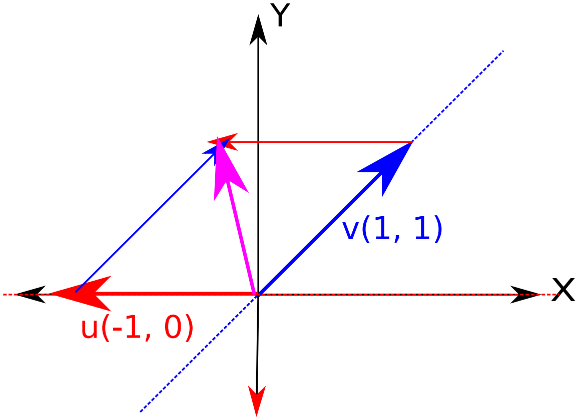
Column Vector Walking Interpretation

$$\begin{bmatrix} | & | & \vdots & | \\ \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_N \\ | & | & \vdots & | \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix} = \sum_{i=1}^{N} a_i \vec{v}_i$$

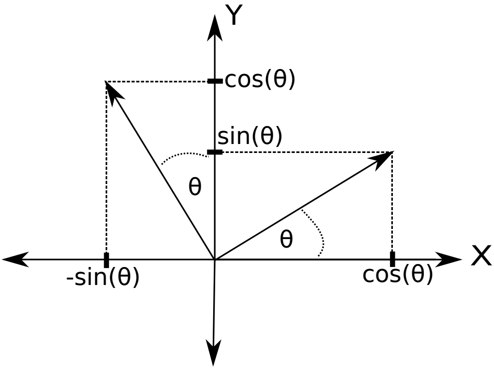
Linear combination of column vectors!

Column Vector Walking Interpretation

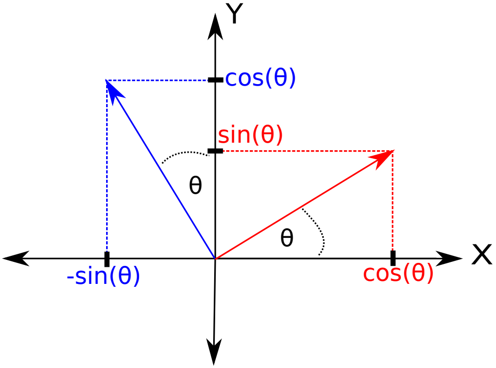
$$\begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} u + \begin{bmatrix} 1 \\ 1 \end{bmatrix} v$$



2D Rotation Matrix Design



2D Rotation Matrix Design



$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

2D Rotation Matrix: Examples

Translation Matrix

$$f((x, y)) = (x + a, y + b)$$

$$\mathbb{R}^2 \to \mathbb{R}^2$$

$$\begin{bmatrix} - & - \\ - & - \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + a \\ y + b \end{bmatrix}$$

??

Homogenous Coordinates

Pure translation with homogenous coordinates

$$\begin{bmatrix} 1 & 0 & T_x \\ 0 & 1 & T_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} T_x \\ T_y \end{bmatrix} \\ 1 \end{bmatrix}$$

Homogenous Coordinates

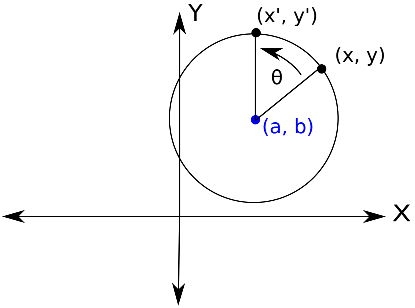
General 2D transformation + translation

$$\begin{bmatrix} a & b & T_x \\ c & d & T_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} T_x \\ T_y \end{bmatrix} \\ 1 \end{bmatrix}$$

We have some extra baggage, but we have more freedom now

Group Raffle Point Question

Write down a matrix which rotates a vector around a point



Formulas to help you

$$T_{(x,y)} = \begin{bmatrix} 1 & 0 & T_x \\ 0 & 1 & T_y \\ 0 & 0 & 1 \end{bmatrix}, R_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Javascript Sphere Plotting