Lecture 6: Normal Transformations, 3D Transformations, Euler Angles

COMPSCI/MATH 290-04

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2/2/2016

Announcements

- Mini Assignment 2 Out, Due Next Monday 11:55 PM
- Online notes coming soon... (for now slides)

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Column Vector Walking Interpretation

$$\begin{bmatrix} | & | & \vdots & | & \vdots & | \\ \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_k & \dots & \vec{v}_N \\ | & | & \vdots & | & \vdots & | \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ a_k \\ \vdots \\ 0 \end{bmatrix} = a_k \vec{v}_k$$

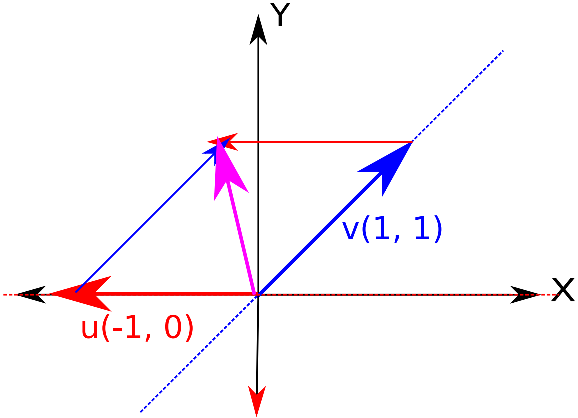
Column Vector Walking Interpretation

$$\begin{bmatrix} | & | & \vdots & | \\ \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_N \\ | & | & \vdots & | \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix} = \sum_{i=1}^{N} a_i \vec{v}_i$$

Linear combination of column vectors!

Column Vector Walking Interpretation

$$\begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} u + \begin{bmatrix} 1 \\ 1 \end{bmatrix} v$$



Linear Function To Matrix Proof

- Define a linear function $L : x \in \mathbb{R}^N \to y \in \mathbb{R}^M$

Linear Function To Matrix Proof

- Define a linear function $L : x \in \mathbb{R}^N \rightarrow y \in \mathbb{R}^M$
- Let $L^j = L\left(\left\{ \begin{array}{cc} 1 & k = j \\ 0 & \text{otherwise} \end{array} \right\}, k = 1 \text{ to } N\right)$

Linear Function To Matrix Proof

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- Let $L^j = L\left(\left\{\begin{array}{cc} 1 & k = j \\ 0 & \text{otherwise} \end{array}\right\}, k = 1 \text{ to } N\right)$

$$\begin{bmatrix} | & | & \vdots & | \\ L^1 & L^2 & \ldots & L^N \\ | & | & \vdots & | \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix} = \sum_{i=1}^{N} a_i L^i$$

Linear Function To Matrix Proof

- Define a linear function $L : x \in \mathbb{R}^N \rightarrow y \in \mathbb{R}^M$
- Let $L^j = L\left(\left\{ \begin{array}{cc} 1 & k = j \\ 0 & \text{otherwise} \end{array} \right\}, k = 1 \text{ to } N\right)$

$$\begin{bmatrix} | & | & \vdots & | \\ L^1 & L^2 & \dots & L^N \\ | & | & \vdots & | \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix} = \sum_{i=1}^{N} a_i L^i$$

- Recall that for a linear function: $L(ax + by) = aL(x) + bL(y)$

Linear Function To Matrix Example

$$f(x, y, z) = (x + y + 2z, 3x - 2y)$$

$$f(1, 0, 0) =$$

$$f(0, 1, 0) =$$

$$f(0, 0, 1) =$$

Square Matrix Inverse

$$A^{-1}A = I$$
$$AA^{-1} = I$$

Square Matrix Inverse

$$A^{-1}A = I$$
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Example:

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, A^{-1} = \begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix}$$

Square Matrix Inverse

$$A^{-1}A = I$$

$$AA^{-1} = I$$

Example:

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, A^{-1} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Square Matrix Product Inverse

$$(AB)^{-1} = B^{-1}A^{-1}$$

Matrix Transpose

$$A^T_{ij} = A_{ji}$$

$$A : M \times N \iff A^T : N \times M$$

Matrix Transpose

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$$A : M \times N \iff A^T : N \times M$$

$$A = \begin{bmatrix} | & | & \vdots & | & \vdots & | \\ \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_k & \dots & \vec{v}_N \\ | & | & \vdots & | & \vdots & | \end{bmatrix}$$

Matrix Transpose

$$A^T_{ij} = A_{ji}$$

$$A : M \times N \iff A^T : N \times M$$

$$A = \begin{bmatrix} | & | & \vdots & | & \vdots & | \\ \vec{v_1} & \vec{v_2} & \dots & \vec{v_k} & \dots & \vec{v_N} \\ | & | & \vdots & | & \vdots & | \end{bmatrix}$$

$$A^T = \begin{bmatrix} - & \vec{v_1} & - \\ - & \vec{v_2} & - \\ \vdots & \dots & \vdots \\ - & \vec{v^N} & - \end{bmatrix}$$

Matrix Transpose Example

$$A = \begin{bmatrix} 1 & 2 & -2 & 3 \\ 0 & 4 & 5 & -1 \\ 6 & -4 & 3 & 2 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 0 & 6 \\ 2 & 4 & -4 \\ -2 & 5 & 3 \\ 3 & -1 & 2 \end{bmatrix}$$

Transpose of Product Matrix

$$(AB)^T = B^T A^T$$

Check dimensions!

Rotation Matrices Inverse

$$R_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

Rotation Matrices Inverse

$$R_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

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Rotation Matrices Inverse

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$$R_\theta^{-1} = R_\theta^T$$

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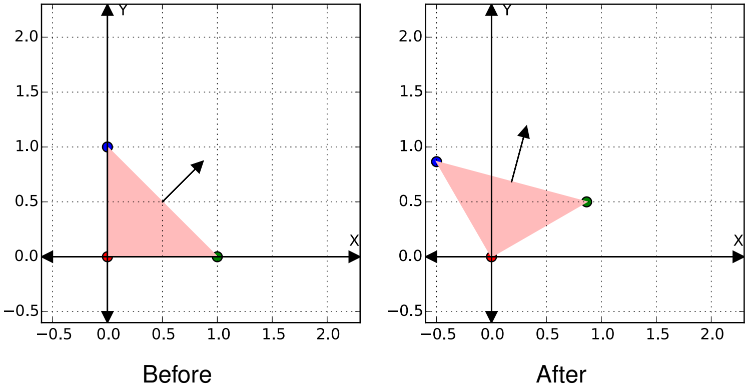
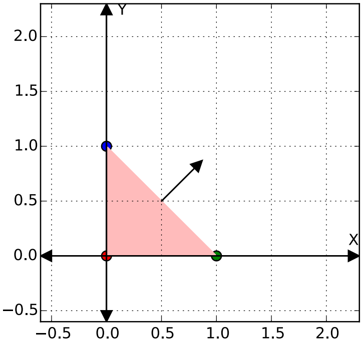
Normal Transformations

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cos(\theta) - y\sin(\theta) \\ x\sin(\theta) + y\cos(\theta) \end{bmatrix}$$



Normal Transformations

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ y \end{bmatrix}$$



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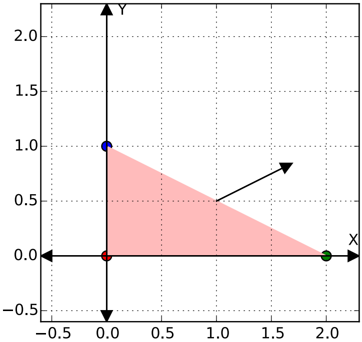
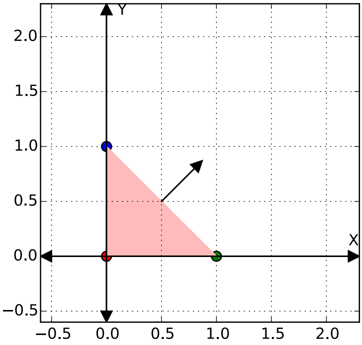


After

Normal Transformations

Uh oh....

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ y \end{bmatrix}$$



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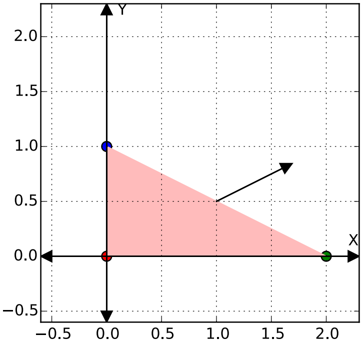
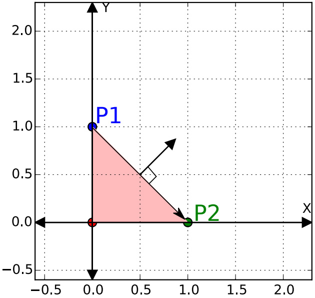


After

Normal Transformation Corrected



Tangent Vector: $\vec{T} = \vec{P_2} - \vec{P_1}$

Normal Vector: $\vec{N}$

Treating as column vectors: $T^T N = 0$

Normal Transformation Corrected

Given Transformation matrix A, transformed tangent vector is

$$AP_2 - AP_1 = A(P_2 - P_1) = AT$$

Normal Transformation Corrected

Given Transformation matrix A, transformed tangent vector is

$$AP_2 - AP_1 = A(P_2 - P_1) = AT$$

Want to find a matrix G s.t. transformed normal GN is orthogonal to AT

$$(AT)^T(GN) = 0$$

Normal Transformation Corrected

$$(AT)^T(GN) = 0$$

Normal Transformation Corrected

$$(AT)^T(GN) = 0$$

$$T^T A^T GN = 0$$

Normal Transformation Corrected

$$(AT)^T(GN) = 0$$

$$T^T A^T GN = 0$$

$$T^T (A^T G) N = 0$$

Normal Transformation Corrected

$$(AT)^T(GN) = 0$$

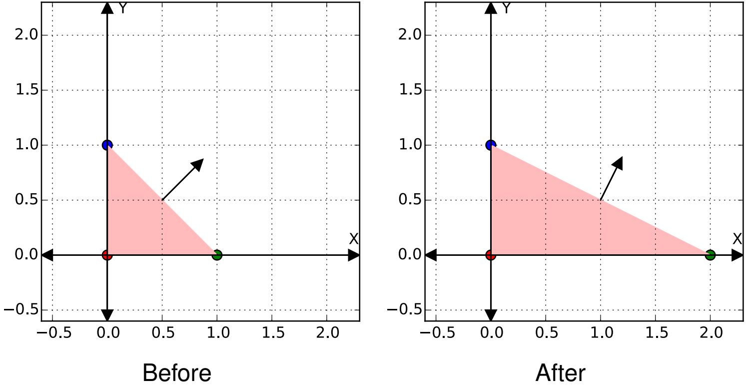
$$T^T A^T G N = 0$$

$$T^T(A^T G)N = 0$$

We know $T^T N = 0$, so $A^T G = I$
Therefore, $G = (A^T)^{-1}$

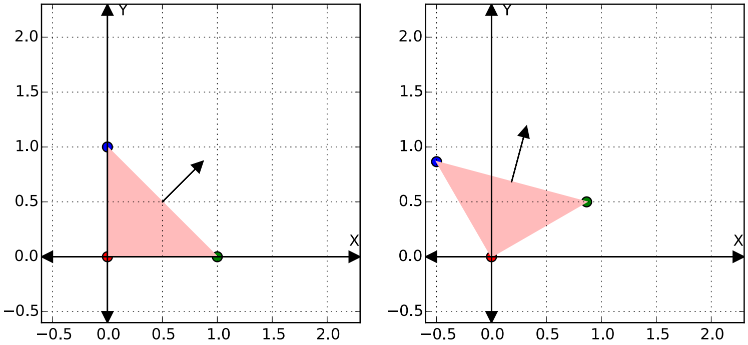
Normal Transformation Corrected

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, G = (A^T)^{-1} = A^{-1} = \begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix}$$



Normal Transformations Corrected

$$A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}, G = (A^T)^{-1} = (A^{-1})^{-1} = A$$

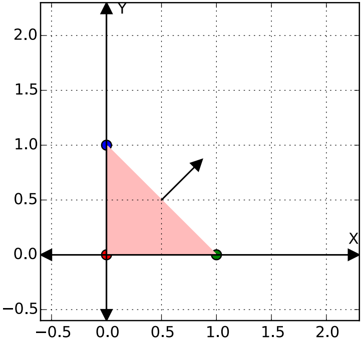


Before

After

Normal Transformations Corrected

$$A = \begin{bmatrix} \frac{3}{2} & 1 \\ 0 & 1 \end{bmatrix}, G = (A^T)^{-1} = \begin{bmatrix} \frac{2}{3} & 0 \\ -\frac{2}{3} & 1 \end{bmatrix}$$



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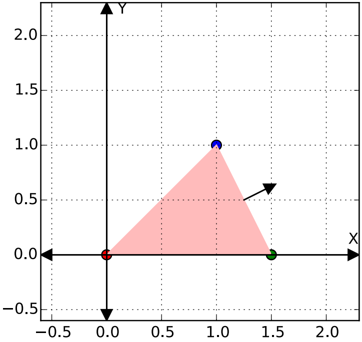

After

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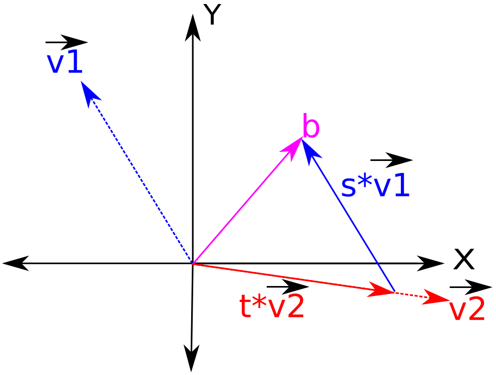
Linear Equations Geometric Interpretation

$$Ax = b$$

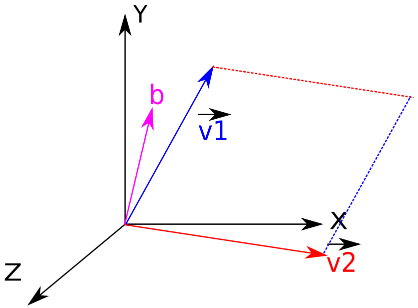
$$\begin{bmatrix} \vec{v1} & \vec{v2} \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} = \vec{b}$$



Linear Equations Geometric Interpretation

$$Ax = b$$

What if A has more rows than columns?

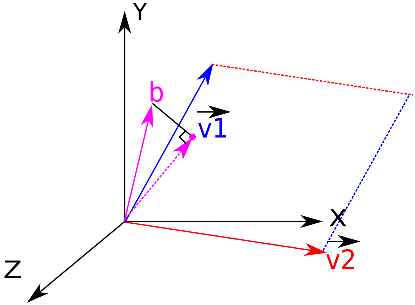


Linear Equations Geometric Interpretation

$$Ax = b$$

What if A has more rows than columns?

$$(A^T A)x = (A^T b)$$



Least Squares solution / Pseudoinverse

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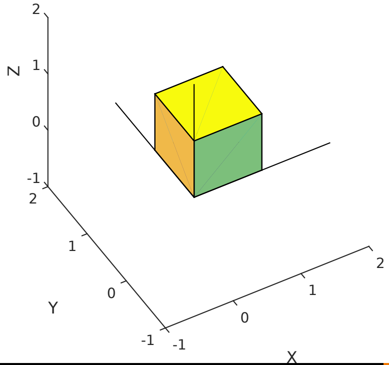
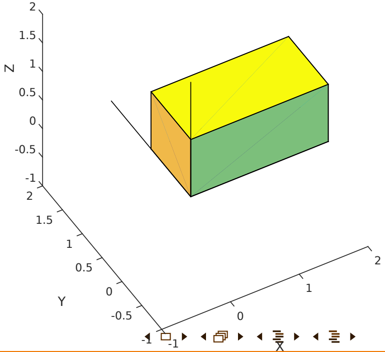
3D Scale X

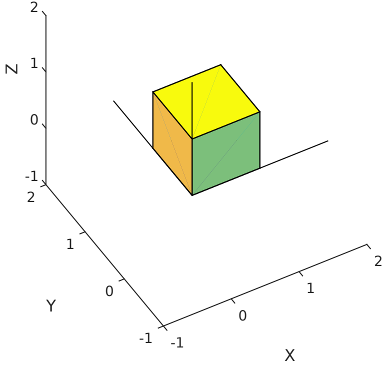
$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2x \\ y \\ z \end{bmatrix}$$





3D Flip XZ

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -x \\ y \\ -z \end{bmatrix}$$



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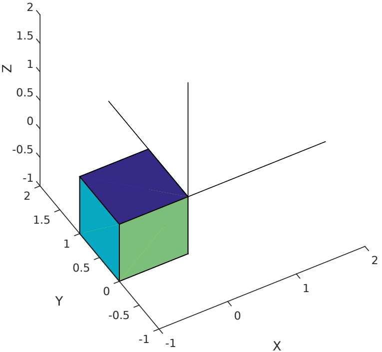
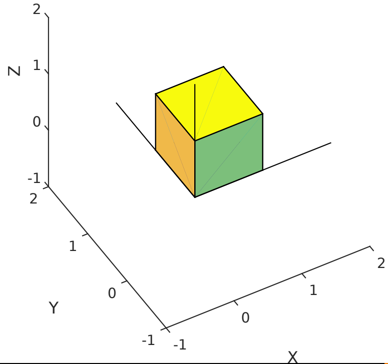
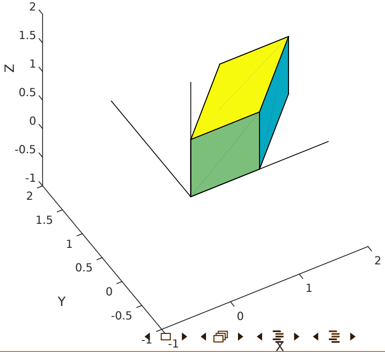


After

X Shear Along Y

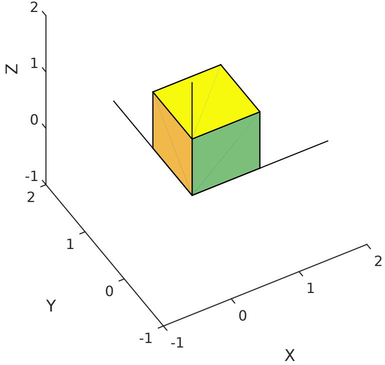
$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+y \\ y \\ z \end{bmatrix}$$





X Shear Along Y and Z

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+y+z \\ y \\ z \end{bmatrix}$$



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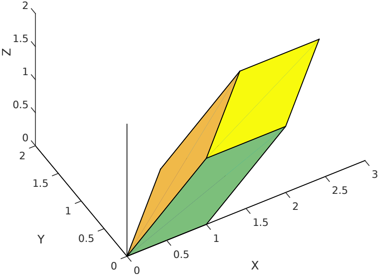
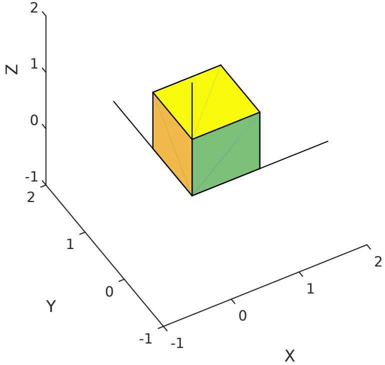


After

X Shear Along Y, Y Shear Along Z

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+y \\ y+z \\ z \end{bmatrix}$$



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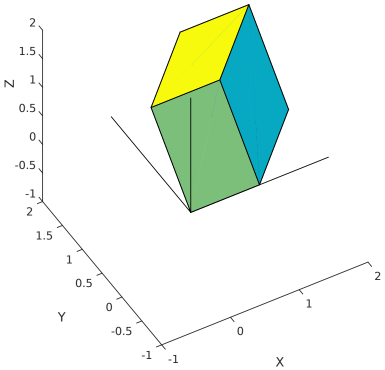


After

X Shear Along Y, Y Shear Along Z

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + y \\ y + z \\ z \end{bmatrix}$$

Interactive Demo

3D Homogenous Coordinates

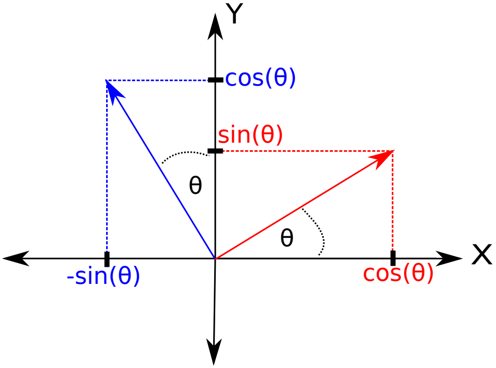
$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} & T_x \\ A_{21} & A_{22} & A_{23} & T_y \\ A_{31} & A_{32} & A_{33} & T_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3D Homogenous Coordinates

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} & T_x \\ A_{21} & A_{22} & A_{23} & T_y \\ A_{31} & A_{32} & A_{33} & T_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A \begin{bmatrix} x \\ 1 \end{bmatrix} = \begin{bmatrix} A^{3\times 3}x + \begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix} \\ 1 \end{bmatrix}$$

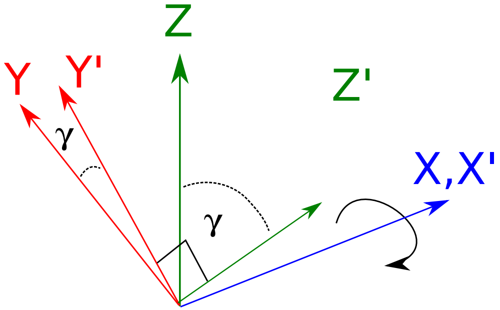
2D Rotation Matrix Design: Review



$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

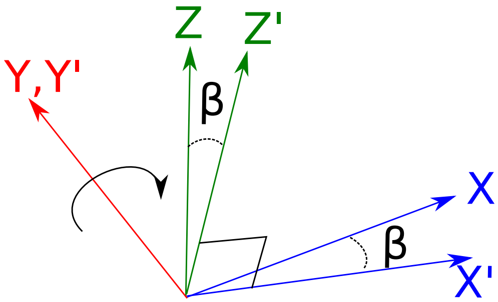
Rotation About X

$$R_X(\gamma) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\gamma) & -\sin(\gamma) \\ 0 & \sin(\gamma) & \cos(\gamma) \end{bmatrix}$$



Rotation About Y

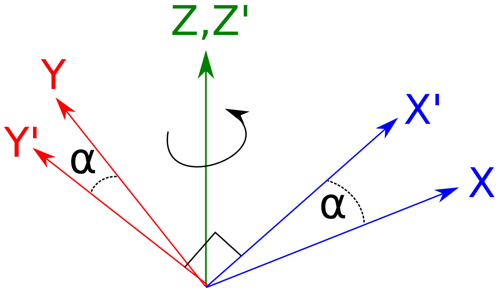
$$R_Y(\beta) = \begin{bmatrix} \cos(\beta) & 0 & \sin(\beta) \\ 0 & 1 & 0 \\ -\sin(\beta) & 0 & \cos(\beta) \end{bmatrix}$$



This one hurts the brain a little

Rotation About Z

$$R_Z(\alpha) = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Just like the normal 2D *XY* rotation

Euler Angles

Can chain these matrices together in any order, such as

$$R_{ZYX} = R_X(\gamma)R_Y(\beta)R_Z(\alpha)$$

$$R_{XYZ} = R_Z(\alpha)R_Y(\beta)R_X(\gamma)$$

Resulting matrix is always *orthogonal*

Euler Angles: Raffle Point Question

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How many degrees of freedom to reach any 3D orientation?
Give me a reason

Euler Angles: Raffle Point Question

How many degrees of freedom to reach any 3D orientation?
Give me a reason

- Each column of the matrix is a unit vector

Euler Angles: Raffle Point Question

How many degrees of freedom to reach any 3D orientation?
Give me a reason

- ▷ Each column of the matrix is a unit vector
- ▷ Every pair of columns is orthogonal.
  In matrix language, $A^T A = I$

Euler Angles

Euler Angles Demo